Relativistic invariant projectors on a complex spinor space and a rule of polarizations summation in a complex bispinor space

H. V. Grushevskaya[1]
Physics Department, Belarusan State University,
4 Nezavisimosti ave., Minsk 220030, BELARUS
E-mail: grushevskaja@bsu.by

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Abstract

Relativistic invariant projectors of states in a complex bispinor space on a complex spinor space are constructed. An expression for sections of bundle with connection on group SU(4) in an explicit form has been obtained. Within the framework of the proposed geometrical approach the rule of summation over polarizations of states in a complex bispinor space has been derived. It has been shown that states in a complex bispinor space always describe a pair of Dirac's particles.

1 Introduction

Technique of covariant projection operators, which for the first time was offered in paper [1], is effectively used to calculate amplitudes of scattering in quantum field theories describing particles with a half-integer spin [2], [3] and vector-bosons [4]. In this method the projection operators are represented as matrixes - diads, for such a construction it is necessary to determine a set of basic bispinors. As a rule, in quantum electrodynamics the basic sets describing states with certain parity [5] are utilized. In the paper we shall determine a basic set of states in a complex bispinor space.

The goal of the paper is to construct relativistic invariant projectors of states in a complex bispinor space on a complex spinor space and to study their properties.

2 Projection operators on a spinor subspace

Let us describe a particle by bispinor wave function, components of which are spinors ξ, $\dot\xi$ [6]:

$$\Psi \sim \begin{pmatrix} \xi \\ \dot\xi \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \dot\xi_1 \\ \dot\xi_2 \end{pmatrix}. \qquad (1)$$

The spinors ξ, $\dot\xi$ are transformed by the representation of the Lorentz group [5]:

$$\xi = \left[\sqrt{\frac{p_0+m}{2m}} + (\vec\sigma\cdot\vec n)\sqrt{\frac{p_0-m}{2m}}\right]\varphi e^{-\imath px}, \qquad (2)$$

$$\dot\xi = \left[\sqrt{\frac{p_0+m}{2m}} - (\vec\sigma\cdot\vec n)\sqrt{\frac{p_0-m}{2m}}\right]\varphi e^{-\imath px}. \qquad (3)$$

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Let us introduce spinor wave functions with defined parity:

$$\Psi_1 = \frac{1}{2}(\xi + \dot{\xi}), \quad \Psi_2 = \frac{1}{2}(\xi - \dot{\xi}), \tag{4}$$

where Ψ_1 is a spinor having positive parity, Ψ_2 is a spinor having negative parity. Substituting eqs. (2) and (3) into eq. (4) we find these spinors in the explicit form:

$$\Psi_1 = \sqrt{\frac{p_0 + m}{2m}} \varphi e^{-\imath px}, \quad \Psi_2 = (\vec{\sigma} \cdot \vec{n}) \sqrt{\frac{p_0 - m}{2m}} \varphi e^{-\imath px}. \tag{5}$$

Introducing into consideration symmetric and antisymmetric spinors we can describe a Dirac particle restricting only appropriate spinor spaces being subspaces of the bispinor space. It means that the bispinor space is divided into two spinor subspaces: symmetric and antisymmetric spinor subspaces $\mathcal{H}^s$, $\mathcal{H}^a$:

$$\mathcal{H} = \mathcal{H}^s \oplus \mathcal{H}^a. \tag{6}$$

Therefore the bispinor wave function in $\mathcal{H}^s$ should be chosen in the form:

$$\begin{aligned} \Psi^- &= \frac{1}{\sqrt{2m}} \begin{pmatrix} \sqrt{p_0 + m}\varphi_{\lambda_s} \\ (\vec{\sigma} \cdot \vec{n})\sqrt{p_0 - m}\varphi_{\lambda_a} \end{pmatrix} e^{-\imath px} = \\ = \sqrt{\frac{p_0 + m}{2m}} &\begin{pmatrix} \varphi_{\lambda_s} \\ (\vec{\sigma} \cdot \vec{p}/(p_0 + m))\varphi_{\lambda_a} \end{pmatrix} e^{-\imath px} = \begin{pmatrix} u^{\lambda_s}(p) \\ u^{\lambda_a}(p) \end{pmatrix} e^{-\imath px}. \end{aligned} \tag{7}$$

Here the label $s(a)$ refers to a spinor with positive (negative) parity. Normalization condition for a bispinor u^{λ_s} immediately follows from the spinor form:

$$\begin{aligned} \overline{u}_\alpha^{\lambda_s}(p) u_\alpha^{\lambda'_s}(p) &= \tfrac{1}{2} \left(\sqrt{p_0 + m}\varphi^\dagger_{\lambda_s}, -\varphi^\dagger_{\lambda_s}(\vec{\sigma} \cdot \vec{n})\sqrt{p_0 - m} \right) \begin{pmatrix} \sqrt{p_0 + m}\varphi_{\lambda'_s} \\ (\vec{\sigma} \cdot \vec{n})\sqrt{p_0 - m}\varphi_{\lambda'_s} \end{pmatrix} \\ &= (p_0 + m - p_0 + m)\delta_{\lambda_s \lambda'_s}/2m = \delta_{\lambda_s \lambda'_s}. \end{aligned} \tag{8}$$

As usual a summation is understood by twice meeting indexes. Since the states with a given polarization can be considered as a basis of two-dimensional spinor space, choosing in space-time tetrades s^τ_μ, $\tau = 1, \ldots, 4$ in the bispinor space the basis will consist of states which are determined by the equations:

$$\left(\frac{1}{2}\gamma_5 \not{s}^\tau - \lambda \right) u^\tau = 0. \tag{9}$$

Let us choose as s^τ_μ that have three zero components from four ones in the rest. Then the basis of bispinor space in the rest has to be chosen by the following form

$$u_\alpha \left(p, s^{1(2)}\right) = \frac{1}{\sqrt{2m}} \begin{pmatrix} \sqrt{p_0 + m}\varphi_{\frac{1}{2}(-\frac{1}{2})} \\ 0 \end{pmatrix}$$

$$u_\alpha \left(p, s^{3(4)}\right) = \frac{1}{\sqrt{2m}} \begin{pmatrix} 0 \\ (\vec{\sigma} \cdot \vec{n})\sqrt{p_0 - m}\varphi_{\frac{1}{2}(-\frac{1}{2})} \end{pmatrix}, \; p_0 = m. \tag{10}$$

Here one took into account that if a center-of-mass of a system is the rest reference system then the component p_0 of four-dimensional velocity p can be considered as the energy $E = p_0$ of the particle in rest: $E = p_0 = m$.

One introduces projection operators $P(s)$ with the help of relation

$$P(\lambda) \equiv P(\vec{s}) = \frac{1 + \vec{\sigma} \cdot \vec{n}}{2}. \tag{11}$$

The operator $P(s)$ (11) in relativistic invariant form is determined by the expression:

$$P(s) \equiv \frac{1 + \gamma_5 \not{s}}{2}. \tag{12}$$

It follows from here that the helicity $\vec{\sigma} \cdot \vec{n}$ of a particle along direction $\vec{s}$ coincides with the direction $\vec{n}$ up to a sign:

$$\vec{s} = \pm \vec{n}. \tag{13}$$

A summation of projection operator over helicities gives unity

$$\sum_{\pm |\vec{s}|} P(\vec{s}) = 1, \tag{14}$$

as it should be.

It is evidently that the spinor description is unsatisfactory because there is no basic antisymmetric spinors in range of positive values for parameter p_0. Let us assume that the parameter p_0 takes on the values:

$$|p_0| \geq m. \tag{15}$$

Let us make the change $p_0 \to -p_0$ in the range of negative values for parameter p_0. Then one can find basic antisymmetric spinors $u^+_\alpha(p, s^\tau)$, $\tau = 1, \ldots, 4$ [5]:

$$\begin{gathered} u^+_\alpha\left(p, s^{1(2)}\right) = \frac{\pm \imath}{\sqrt{2m}} \begin{pmatrix} \sqrt{p_0 - m}\,\varphi_{\frac{1}{2}(-\frac{1}{2})} \\ 0 \end{pmatrix}, \\ u^+_\alpha\left(p, s^{3(4)}\right) = \frac{\pm \imath}{\sqrt{2m}} \begin{pmatrix} 0 \\ (\vec{\sigma} \cdot \vec{n})\sqrt{p_0 + m}\,\varphi_{\frac{1}{2}(-\frac{1}{2})} \end{pmatrix}. \end{gathered} \tag{16}$$

It follows from here that the basic antisymmetric spinors are imaginary. Hence, we have complexified the spinor space by the change $p_0 \to -p_0$. Really, in a moving reference frame the change $p_0 \to -p_0$ becomes the change $p_0 \to -p_0$, $\vec{p} \to -\vec{p}$. As a result of this change, the symmetric spinor Ψ^- (7) becomes an antisymmetric imaginary spinor Ψ^+:

$$\Psi^+ = \frac{\pm \imath}{\sqrt{2m}} \begin{pmatrix} \sqrt{p_0 - m}\,\varphi_{\lambda_s} \\ (\vec{\sigma} \cdot \vec{n})\sqrt{p_0 + m}\,\varphi_{\lambda_a} \end{pmatrix} e^{\imath p x} = \begin{pmatrix} u^{\lambda_s}(-p) \\ u^{\lambda_a}(-p) \end{pmatrix} e^{\imath p x}. \tag{17}$$

Now taking into account that only two nonzero basic spinors are included into a basic set (10), (16) we with the help of projection operator technique are able to write a rule of summation over spinor polarizations in the relativistic invariant form:

$$\begin{aligned} \sum_{\lambda = \pm 1/2} u^\lambda_\alpha(p) \overline{u}^\lambda_\beta(p) = \frac{1}{2m} \sum_{\lambda = \lambda_s, \lambda_a} (\not{p} + m)_{\alpha\gamma} u^\lambda_\gamma \overline{u}^\lambda_\beta &= \sum_{\lambda = \pm 1/2} \sum_{\Phi} \left(\frac{\not{p} + m}{2m} \right)_{\alpha\kappa} P_{\kappa\delta}(\lambda) |\Phi\rangle \gamma_0 \langle \Phi | P_{\delta\beta}(\lambda) \\ &= \sum_{\tau = 1}^{2} \sum_{\Phi'} \left(\frac{\not{p} + m}{2m} \right)_{\alpha\kappa} P_{\kappa\delta}(s^\tau) |\Phi'\rangle \gamma_0 \langle \Phi' | P_{\delta\beta}(s^\tau) \end{aligned} \tag{18}$$

Now, one will take into account a unity decomposition over vectors $|\Phi'\rangle$:

$$\hat{1} = \sum_{\Phi'} |\Phi'\rangle\gamma_0\langle\Phi'|, \tag{19}$$

where the matrix - diad $|\Phi'\rangle\gamma_0\langle\Phi'| \equiv \left(u^+_\alpha \cdot \gamma_0 (u^{+T}_\alpha)^\dagger\right)\Big|_{p_0=-m}$, u^+_α is defined by the formula (16), $\varphi^T_{-1/2} = (0\ 1)$, $\varphi^T_{+1/2} = (1\ 0)$, $\dagger$ denotes a complex conjugation, $\hat{1}$ is the unity operator. Then eq. (18) can be rewritten as

$$\sum_{\lambda=\pm 1/2} u^\lambda_\alpha(p)\overline{u}^\lambda_\beta(p) = \frac{1}{2m}\sum_{\lambda=\lambda_s,\lambda_a}(\not{p}+m)_{\alpha\gamma}u^\lambda_\gamma\overline{u}^\lambda_\beta = \sum_{\tau=1}^{2}\left(\frac{\not{p}+m}{2m}\right)_{\alpha\kappa} P_{\kappa\beta}(s^\tau)\hat{1} = \frac{1}{2m}(\not{p}+m)_{\alpha\beta}. \tag{20}$$

In a similar manner as for the symmetric spinors we find a sum of antisymmetric spinor products over polarizations:

$$\sum_{\lambda_s=\pm 1/2} u^{\lambda_s}_\alpha(-p)\overline{u}^{\lambda_s}_\beta(-p) = \frac{1}{2m}\sum_{\lambda=\lambda_s,\lambda_a}(\not{p}-m)_{\alpha\kappa}u^\lambda_\kappa\overline{u}^\lambda_\beta =$$

$$= \sum_{\tau=3}^{4}\sum_{\Phi'}\left(\frac{\not{p}-m}{2m}\right)_{\alpha\kappa} P_{\kappa\delta}(s^\tau)(\pm\imath)|\Phi'\rangle\gamma_0\langle\Phi'|(\pm\imath)P_{\delta\beta}(s^\tau)$$

$$= \sum_{\tau=3}^{4}\left(\frac{\not{p}-m}{2m}\right)_{\alpha\kappa} P_{\kappa\beta}(s^\tau)(-\hat{1}) = \frac{1}{2m}(m-\not{p})_{\alpha\beta}. \tag{21}$$

3 Projection operators on bispinor space

Let us examine a bispinor space generated by the bispinors Ψ defined by the expression (1), upper components of which are the spinors ξ defined by the expression (2), and lower components are $\dot{\xi}$ defined by the expression (3).

Let s^τ_μ, $\tau = 1,\ldots,4$ in eq. (9) has three zero components from four ones. Then, in rest reference system one has to choose as a basis of bispinor space the following bispinors

$$u_\alpha\left(p_0, \vec{p}=0, s^{1(2)}\right) = \frac{1}{\sqrt{2m}}\begin{pmatrix}\sqrt{p_0+m}\varphi_{\frac{1}{2}(-\frac{1}{2})}\\ 0\end{pmatrix}, \tag{22}$$

$$u_\alpha\left(p_0, \vec{p}=0, s^{3(4)}\right) = \frac{1}{\sqrt{2m}}\begin{pmatrix}0\\ \sqrt{p_0+m}\varphi_{\frac{1}{2}(-\frac{1}{2})}\end{pmatrix}; \tag{23}$$

where bispinors $u_\alpha\left(p, s^{1(2)}\right)$ are transformed over the representation ξ, and bispinors $u_\alpha\left(p, s^{3(4)}\right)$ are transformed over the representation $\dot{\xi}$. The tetrad s is called the helicity also. From the expressions (2), (3), and (13) describing $u_\alpha(p, s^\tau)$ in an arbitrary reference frame one has the equality:

$$\left(\vec{\sigma}\cdot\vec{s}^{1(2)}u_\alpha\right)(p_0, \vec{p}=0, s^{1(2)}) = \pm\kappa u_\alpha(p_0, \vec{p}=0, s^{1(2)}), \tag{24}$$

$$\left(\vec{\sigma}\cdot\vec{s}^{3(4)}u_\alpha\right)(p_0, \vec{p}=0, s^{3(4)}) = \pm(-\kappa) u_\alpha(p_0, \vec{p}=0, s^{3(4)}), \tag{25}$$

where κ is determined by the expression

$$\kappa = \sqrt{\frac{p_0-m}{p_0+m}} \tag{26}$$

One concludes from here that κ must be equal to ± 1: $\kappa = \pm 1$. But $\kappa(p_0 = m) \neq 1$. It means that although the bispinors with helicity s^τ, $\tau = 1, 2$ under the condition $p_0 = m$, are solutions of the Dirac's equation but they are not solutions of equations for eigenvalues of relativistic spin of Dirac particle. Hence, in the region $p_0 \geq m$ for values of the parameter p_0 there do not exist basis of the bispinor space. To find the basis and, hence, to construct the bispinor space one extends the region for values of the parameter p_0:

$$|p_0| \geq 0 \tag{27}$$

Let us consider the region $|p_0| \leq m$. According to the expression (26) one has to make the change $\vec{n} \to \imath\vec{n}$ in this region. From here it follows that one can define a complex bispinor $\breve{u}$ as

$$\breve{u} = \left(\begin{array}{c} \left(\sqrt{\frac{p_0+m}{2m}} + \imath(\vec{\sigma}\cdot\vec{n})\sqrt{\frac{p_0-m}{2m}}\right)\varphi_{\lambda_+} \\ \left(\sqrt{\frac{p_0+m}{2m}} - \imath(\vec{\sigma}\cdot\vec{n})\sqrt{\frac{p_0-m}{2m}}\right)\varphi_{\lambda_-} \end{array} \right) = \left(\begin{array}{c} \breve{u}^{\lambda_+} \\ \breve{u}^{\lambda_-} \end{array} \right). \tag{28}$$

Let us find its Dirac conjugation $\overline{\breve{u}}$. To do it one utilizes a probabilistic interpretation of scalar product of bispinor wave functions $\breve{u}$, $\overline{\breve{u}}$. It means that we have to compose a conserved quantity. Evidently, a product of spinors ξ, $\dot{\xi}$ is a relativistic invariant because they are transformed on different representations of Lorentz group. Really, we have

$$\dot{\xi}'^{\dagger}\xi' = \dot{\xi}^{\dagger} e^{-\frac{\chi}{2}(\vec{\sigma}\cdot\vec{n})} e^{\frac{\chi}{2}(\vec{\sigma}\cdot\vec{n})}\xi = \dot{\xi}^{\dagger}\xi. \tag{29}$$

Writing ξ, $\dot{\xi}$ through linear combinations of symmetric and antisymmetric spinors Ψ_1, Ψ_2, one gets that

$$\dot{\xi}^{\dagger}\xi = (\Psi_1^{\dagger} - \Psi_2^{\dagger}, \Psi_1 + \Psi_2) = \Psi_1^{\dagger}\Psi_1 - \Psi_2^{\dagger}\Psi_2 = \overline{u}^{\lambda}(p)u^{\lambda}(p). \tag{30}$$

It follows from here that $\overline{u}^{\lambda}(p)u^{\lambda}(p)$ is the relativistic invariant. In the probabilistic interpretation the product $\overline{\breve{u}}^{\lambda}(p)\breve{u}^{\lambda}(p)$ plays the same role in bispinor space as the product $\overline{u}^{\lambda}(p)u^{\lambda}(p)$ in spinor space. Hence, the product $\overline{\breve{u}}^{\lambda}(p)\breve{u}^{\lambda}(p)$ must be relativistic invariant. To satisfy this condition, in accordance with the expression (30) the probability $\overline{\breve{u}}^{\lambda}(p)\breve{u}^{\lambda}(p)$ must be represented as a combination of terms $\dot{\xi}^{\dagger}\xi$. It means that the conjugated complex bispinor $\overline{\breve{u}}$ can be defined as

$$\overline{\breve{u}} = \left[\gamma_5 \left(\begin{array}{c} \varphi^{\dagger}\left(\sqrt{\frac{p_0+m}{2m}} - \imath(\vec{\sigma}\cdot\vec{n})\sqrt{\frac{p_0-m}{2m}}\right) \\ \varphi^{\dagger}\left(\sqrt{\frac{p_0+m}{2m}} + \imath(\vec{\sigma}\cdot\vec{n})\sqrt{\frac{p_0-m}{2m}}\right) \end{array} \right)\right]^T. \tag{31}$$

Let us calculate a norm of bispinor $\breve{u}$ introduced with the help of the relation (28):

$$\overline{\breve{u}}_\alpha\breve{u}_\alpha = \left(\begin{array}{c} \varphi^{\dagger}_{\lambda_+}\left(\sqrt{\frac{p_0+m}{2m}} + \imath(\vec{\sigma}\cdot\vec{n})\sqrt{\frac{p_0-m}{2m}}\right) \\ \varphi^{\dagger}_{\lambda_-}\left(\sqrt{\frac{p_0+m}{2m}} - \imath(\vec{\sigma}\cdot\vec{n})\sqrt{\frac{p_0-m}{2m}}\right) \end{array} \right)^T \left(\begin{array}{c} \left(\sqrt{\frac{p_0+m}{2m}} + \imath(\vec{\sigma}\cdot\vec{n})\sqrt{\frac{p_0-m}{2m}}\right)\varphi_{\lambda_+} \\ \left(\sqrt{\frac{p_0+m}{2m}} - \imath(\vec{\sigma}\cdot\vec{n})\sqrt{\frac{p_0-m}{2m}}\right)\varphi_{\lambda_-} \end{array} \right) = 2. \tag{32}$$

Now let us show that in the built space of complex bispinors there exist a basis satisfying the system from the Dirac equation and the equation for relativistic spin which describes a Dirac particle. Hermitian conjugated Dirac equation has the form:

$$\breve{u}^{\dagger}(\not{p}^{\dagger} - m) = 0, \tag{33}$$

where

$$\not{p}^{\dagger} = (\gamma_0 p_0 - \vec{\gamma}\cdot\vec{p})^{\dagger} = -(\gamma_0 p_0 + \vec{\gamma}\cdot\vec{p}) \tag{34}$$

owing to hermicity of γ_0 and anti-hermicity of γ_i. Let us multiply eq. (33) by γ_5 from right. Then taking into account that the matrix γ_5 is represented in the form: $\gamma_5 = \imath\gamma_0\gamma_1\gamma_2\gamma_3$, one gets

$$\breve{u}^\dagger(\gamma_0 p_0(\imath\gamma_0\gamma_1\gamma_2\gamma_3) + \vec{\gamma}\cdot\vec{p}(\imath\gamma_0\gamma_1\gamma_2\gamma_3) + m) = 0. \tag{35}$$

The matrix γ_5 is a pseudoscalar: $\gamma_5 = -\frac{\imath}{4!}\epsilon_{\mu\nu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho = \imath\gamma_0\gamma_1\gamma_2\gamma_3$. Using anticommutation relations for the γ - matrixes :

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \tag{36}$$

one can rewrite eq.(35) as

$$\begin{aligned}\breve{u}^\dagger\Big(-\frac{\imath}{4!}\epsilon_{\nu\sigma\rho\mu}\gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\mu\gamma_0 p_0 - \frac{\imath}{4!}\epsilon_{\nu\mu\sigma\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\gamma_1 p_1 + \frac{\imath}{4!}\epsilon_{\sigma\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\gamma_2 p_2 \\ -\frac{\imath}{4!}\epsilon_{\rho\mu\nu\sigma}\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\gamma_3 p_3 + m\Big) = 0.\end{aligned} \tag{37}$$

Therefore the new field $\overline{\breve{u}}$ satisfies the Dirac equation

$$\overline{\breve{u}}(\not{p} + m) = 0. \tag{38}$$

Let us put $p_0 = 0$. Then the basic bispinors $u^\tau_\alpha(0)$, $\tau = 1, \ldots, 4$ are defined in the following way:

$$\breve{u}^1(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \breve{u}^2(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \breve{u}^3(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ \breve{u}^4(0) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{39}$$

We see that the center-of-mass system ($p_0 = m,\ \vec{p} = 0$) is excluded as the rest frame. Changing s for $\pm\imath n$ with the help of the expression (13), the system of equations (24, 25) can be rewritten in the form

$$(\vec{\sigma}\cdot\vec{n}\breve{u}^{1(2)}_\alpha)(0) = \breve{u}^{1(2)}_\alpha(0), \tag{40}$$

$$(\vec{\sigma}\cdot\vec{n}\breve{u}^{3(4)}_\alpha)(0) = -\breve{u}^{3(4)}_\alpha(0). \tag{41}$$

Hence, we prove that the projection of particle spin in built complex bispinor space is described by the operator of Dirac particle spin projection.

Using the technique of calculations with the help of projection operators determined in our case by the expressions (12) and taking into account that all 4 basic bispinors are nonzero we get a rule of summation over polarization in the form

$$\sum_{\lambda_+=\pm 1/2}\breve{u}^{\lambda_+}_\alpha(p)\overline{\breve{u}}^{\lambda_+}_\beta(p) = \sum_{\vec{s}=\{\pm\vec{n}\}}\sum_\Phi P_{\alpha\delta}(\vec{s})|\Phi\rangle\gamma_5\langle\Phi|P_{\delta\beta}(\vec{s}) = \sum_{\tau=1}^{4}\sum_{\Phi'} P_{\alpha\delta}(s^\tau)|\Phi'\rangle\gamma_5\langle\Phi'|P_{\delta\beta}. \tag{42}$$

According to the expression (39) we have

$$\sum_{\Phi'}|\Phi'\rangle\gamma_5\langle\Phi'| = \frac{\hat{1}}{2} \tag{43}$$

because the matrix - diad $|\Phi'\rangle\gamma_5\langle\Phi'|$ is equal to

$$|\Phi'\rangle\gamma_5\langle\Phi'| \equiv \Big\{\big[\imath\ \Im m\ \breve{u}^+_\alpha\big]\cdot\gamma_5\Big[\imath\ \Im m\ \big((\breve{u}^+_\alpha)^T\big)^\dagger\Big]\Big\}\Big|_{p_0=0}, \tag{44}$$

$\breve{u}^+_\alpha$ is defined by the formula (28).

We see that due to the expression (39), the sum over polarizations (42) is built on a basic half-set: $\sum_\Phi |\Phi\rangle\gamma_5\langle\Phi| = \frac{\hat{1}}{2}$. Besides, the norm of complex spinor is two times less than the norm of complex bispinor. Hence, the probability to find a quantum object described by the complex bispinor is two times larger than for a quantum object described by the complex spinor. From here it follows physical sense of the basis with $p_0 = 0$. The built basis describes a pair "particle-antiparticle". As we saw for the particle $p_0 = E$, for the antiparticle $p_0 = -E$. Therefore the parameter p_0 of the pair "particle-antiparticle" equals to zero.

Now, for the basic bispinors $\breve{u}^{1(2)}_\alpha(p)$, we can rewrite the sum (42) of bispinors $\breve{u}^{\lambda_+}_\alpha(p)$ transformed by representation ξ without point in the relativistic invariant form

$$\sum_{\lambda_+=\pm 1/2} \breve{u}^{\lambda_+}_\alpha(p)\overline{\breve{u}}^{\lambda_+}_\beta(p) = \frac{1}{2m}\sum_{\tau=1}^{4}(\not{p}+m)_{\alpha\gamma}\breve{u}^\tau_\gamma\overline{\breve{u}}^\tau_\beta/2$$
$$= \frac{1}{2m}\sum_{\tau=1}^{4}(\not{p}+m)_{\alpha\gamma}P_{\gamma\beta}(s^\tau)/2 = \frac{1}{2}\left(\frac{\not{p}+m}{2m}\sum_{\tau=1}^{4}P(s^\tau)\right)_{\alpha\beta}. \tag{45}$$

Since $P(s)$ is a projection operator one obtains

$$\sum_{\tau=1}^{4} P(s^\tau)/2 = 1. \tag{46}$$

Substituting the expression (46) into the expression (45), we get the rule of summation over polarizations λ_+:

$$\sum_{\lambda_+=\pm 1/2} \breve{u}^{\lambda_+}_\alpha(p)\overline{\breve{u}}^{\lambda_+}_\beta(p) = \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta}. \tag{47}$$

Similarly for the basic bispinors $\breve{u}^{3(4)}_\alpha(p)$, one gets the sum over polarizations for bispinors transformed by representation $\dot{\xi}$:

$$\sum_{\lambda_-=\pm 1/2} \breve{u}^{\lambda_-}_\alpha(p)\overline{\breve{u}}^{\lambda_-}_\beta(p) = \sum_{\vec{s}=\{\pm\vec{n}\}}\sum_{\Phi} P_{\alpha\delta}(\vec{s})(\imath)|\Phi\rangle\gamma_5\langle\Phi|(\imath)P_{\delta\beta}(\vec{s}) =$$
$$= -\sum_{\tau=1}^{4}\sum_{\Phi'} P_{\alpha\delta}(s^\tau)|\Phi'\rangle\gamma_5\langle\Phi'|P_{\delta\beta} = -\frac{\hat{1}}{2}\sum_{\tau=1}^{4}\breve{u}^\tau_\alpha(p)\overline{\breve{u}}^\tau_\beta(p) =$$
$$= -\frac{1}{2m}\sum_{\tau=1}^{4}(\not{p}-m)_{\alpha\gamma}\breve{u}^\tau_\gamma\overline{\breve{u}}^\tau_\beta/2 = -\frac{1}{2m}\sum_{\tau=1}^{4}(\not{p}-m)_{\alpha\gamma}P_{\gamma\beta}(s^\tau)/2 = \left(\frac{m-\not{p}}{2m}\right)_{\alpha\beta}. \tag{48}$$

The projection operators $\breve{u}^\lambda_\alpha(p)\overline{\breve{u}}^\lambda_\beta(p)$ constitute a total set. Indeed, it is easy to show that

$$\sum_{\lambda=\{\lambda_\pm\}} \breve{u}^\lambda_\alpha(p)\overline{\breve{u}}^\lambda_\beta(p) = \left(\frac{\not{p}+m}{2m}\right)_{\alpha\beta} + \left(\frac{m-\not{p}}{2m}\right)_{\alpha\beta} = \hat{1}. \tag{49}$$

Let us find the projection operators $\pi_{\alpha\beta}(p) \equiv \breve{u}_\alpha(p)\overline{\breve{u}}_\beta(p)$ in a explicit form. To do it, we rewrite the expression (48) in the form:

$$\sum_{\lambda_-=\pm 1/2} \breve{u}^{\lambda_-}_\alpha(p)\overline{\breve{u}}^{\lambda_-}_\beta(p) = -\frac{1}{4m}\sum_{\vec{s}=\{\pm\vec{n}\}}\left[(\not{p}-m)\left(1-\gamma_5\vec{\gamma}\cdot\vec{s}\right)\right]_{\alpha\beta}. \tag{50}$$

Then, we obtain

$$\pi^{\lambda}_{\alpha\beta}(p) \equiv \breve{u}^{\lambda}_{\alpha}(p)\overline{\breve{u}}^{\lambda}_{\beta}(p) = -\frac{1}{4m}\left[(\not p - m)\left(1 - \gamma_5\vec{\gamma}\cdot\vec{s}\right)\right]_{\alpha\beta}. \tag{51}$$

Replacing $p \to -p$, interchanging upper and lower components of bispinors by multiplying on left and on right by matrix γ_5, and using rules of multiplication for matrixes γ_5, γ_μ, the expression (45) becomes an expression of the form:

$$\sum_{\lambda_-=\pm 1/2} \breve{u}^{-\lambda_-}_{\alpha}(-p)\overline{\breve{u}}^{-\lambda_-}_{\beta}(-p) = \frac{1}{4m}\left[(\not p + m)\sum_{\vec{s}=\{\pm\vec{n}\}}\gamma_5\left(1 - \gamma_5\vec{\gamma}\cdot\vec{s}\right)\gamma_5\right]_{\alpha\beta}. \tag{52}$$

It follows from here that there exist yet one type of projection operators which is denoted as $\pi^{-\lambda}_{\alpha\beta}(-p)$:

$$\pi^{-\lambda}_{\alpha\beta}(-p) \equiv \breve{u}^{-\lambda}_{\alpha}(-p)\overline{\breve{u}}^{-\lambda}_{\beta}(-p) = \frac{1}{4m}\left[(\not p + m)\left(1 - (\vec{\gamma}\cdot\vec{s})\gamma_5\right)\right]_{\alpha\beta}. \tag{53}$$

Having in mind the explicit expression, for example, of the projection operator $\breve{u}^{\lambda}_{\alpha}(p)\overline{\breve{u}}^{\lambda}_{\beta}(p)$, we can establish correspondence between the complex spinor and complex bispinor representations by the following way. From the expression (51) for the projection operator and from the fact that complex bispinor describes the fermion pair, it follows that given projector projects on a discrete space with a basis from eigenvectors, eigenvalues of which lay in a gap of energy spectrum of the Dirac's operators. Therefore integer spin $\vec{s}$ can be attributed to these vectors. Hence, physical states are described by such bispinors $\breve{u}^{\lambda}_{\alpha}(p)$, the projection of which equals to zero:

$$(\pi^{\lambda}_{\alpha\beta}\breve{u}_{\beta})(p) = 0. \tag{54}$$

Therefore owing the motion equations the physical states of quantum system must be solutions of the equation as

$$\begin{aligned}(\pi^{\lambda}_{\alpha\beta}\breve{u}_{\beta})(p) = -\frac{1}{4m}\Big[\big((\not p - m)\breve{u}^{\lambda}(p)\big)_{\alpha} - \left((\not p - m)\gamma_5\vec{\gamma}\cdot\vec{s}\right)_{\alpha\beta}\breve{u}^{\lambda}_{\beta}(p)\Big]\\ = \frac{1}{4m}\left((\not p - m)\gamma_5\vec{\gamma}\cdot\vec{s}\right)_{\alpha\beta}\breve{u}^{\lambda}_{\beta}(p) = 0\end{aligned} \tag{55}$$

or solutions of the equation as

$$\begin{aligned}(\pi^{-\lambda}_{\alpha\beta}\breve{u}_{\beta})(-p) = \frac{1}{4m}\Big[\big((\not p + m)\breve{u}^{-\lambda}(-p)\big)_{\alpha} - \left((\not p + m)(\vec{\gamma}\cdot\vec{s})\gamma_5\right)_{\alpha\beta}\breve{u}^{-\lambda}_{\beta}(-p)\Big] =\\ = -\frac{1}{4m}\left((\not p + m)(\vec{\gamma}\cdot\vec{s})\gamma_5\right)_{\alpha\beta}\breve{u}^{-\lambda}_{\beta}(-p) = 0.\end{aligned} \tag{56}$$

From here we get that there exist the following relation between the spinors $u^{\lambda}_{\alpha}(p)$, $v^{\lambda}_{\alpha}(p) \equiv u^{-\lambda}_{\alpha}(-p)$ and the bispinors $\breve{u}^{\lambda}_{\alpha}(p)$, $\breve{u}^{-\lambda}_{\alpha}(-p)$, respectively:

$$u^{\lambda}(p) = \gamma_5(\vec{\gamma}\cdot\vec{s})\breve{u}^{\lambda}(p), \tag{57}$$

$$v^{\lambda}(p) = (\vec{\gamma}\cdot\vec{s})\gamma_5\breve{u}^{-\lambda}(-p). \tag{58}$$

But, the complex description in spinor space is equivalent to a real description in Minkowski space [8]. Now, we can show that gauge-invariant states of system lay in a connection ϖ of bundle associated with principal bundle, a base of which is the Minkowski space $\mathbb{M}$, and the connection is the unitary group $\mathsf{SU}\ (4)$. Indeed, we see from eqs. (55) and (56) that the projection $\pi_{\alpha\beta}$ of fiber for this bundle on the base $\mathbb{M}$ equal to zero gives a motion equation of Dirac's particle.

Further, we consider sections of found bundle.

4 Section of bundle with connection on group SU(4)

Let us examine a complex spinor $|x_\lambda^{R(L)}\rangle$ with a given helicity. Here indexes R, L denote right- and left-helical Dirac's particles, respectively. Since the complex bispinor describes a doubled spinor, one can write in rest the identity:

$$\sum_\mu |\xi_\mu\rangle\gamma_5\langle\,\xi_\mu|\psi\rangle \equiv \sum_\lambda \left(|x_\lambda^L\rangle\gamma_0\langle x_\lambda^L| + |x_\lambda^R\rangle\gamma_0\langle x_\lambda^R|\right)\psi\rangle. \tag{59}$$

Let us show that the identity (59) in a rest reference frame can be written as

$$\sum_\mu |\xi_\mu\rangle\gamma_5\langle\,\xi_\mu|\psi\rangle + \frac{1}{2}\sum_\lambda\left[|\xi_\kappa\rangle\gamma_5\left(\sigma_\lambda^+\right)_{\kappa\nu}\langle\,\xi_\nu|\psi\rangle + |\xi_\kappa\rangle\gamma_5\left(\sigma_\lambda^-\right)_{\kappa\nu}\langle\,\xi_\nu|\psi\rangle\right]$$
$$\equiv \sum_\lambda\left[\frac{1}{2}\left(|x_\lambda^L\rangle\gamma_0\langle x_\lambda^L| + |x_\lambda^R\rangle\gamma_0\langle x_\lambda^R|\right)\psi\rangle + \frac{1}{2}\left(|x_\lambda^L\rangle\gamma_0\langle x_\lambda^L| + |x_\lambda^R\rangle\gamma_0\langle x_\lambda^R|\right)\psi\rangle\right], \tag{60}$$

where matrices $\sigma_\lambda^\pm$ are determined by

$$\sigma_\lambda^+ = \epsilon_{\lambda ij}\sigma_{ij};\ \ \sigma_\lambda^- = \epsilon_{\lambda ji}\sigma_{ij},\ \ \lambda, i, j = 1, 2, 3; \tag{61}$$

indexes κ, μ, ν run over $s, \dot{s}$; $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, γ_μ are Dirac matrices. Hereinafter, we will omit sign $\pm$. Since the skew-symmetric tensor $\epsilon_{\lambda jk}$ appears in the equation for σ_λ, the generalized Pauli matrixes σ_λ are pseudo-vectors.

It follows from the expansion of the wave function $|\psi\rangle$ (60) in a series that the projection P defined by the expression:

$$P|\psi\rangle = \frac{1}{2}\left(|x_\lambda^R\rangle\gamma_0\langle x_\lambda^R| + |x_\lambda^L\rangle\gamma_0\langle x_\lambda^L|\right)\psi\rangle, \tag{62}$$

can be represented in the form

$$P|\psi\rangle = \frac{1}{2}\left(|x_\lambda^R\rangle\gamma_0\langle x_\lambda^R| + |x_\lambda^L\rangle\gamma_0\langle x_\lambda^L|\right)\psi\rangle = \frac{1}{2}\left\{\sum_\mu |\xi_\mu\rangle\gamma_5\langle\,\xi_\mu|\psi\rangle + \sum_\lambda |\xi_\kappa\rangle\gamma_5\left(\sigma_\lambda\right)_{\kappa\nu}\langle\,\xi_\nu|\psi\rangle\right\}. \tag{63}$$

Let us prove that the projector (63) selects states with a given orientation. The right side of the expression (63) in the representation $\{|\xi_i\rangle\}_i$ is written as

$$P\langle\xi_\lambda|\psi\rangle = \frac{1}{2}\left\{1 + \sum_\lambda\langle\xi_\lambda|\xi_\kappa\rangle\gamma_5\left(\sigma_\lambda\right)_{\kappa\nu}\right\}\langle\xi_\nu|\psi\rangle \tag{64}$$

We can introduce a four-vector $s^\mu = \{0, s^\lambda\}$, $s^\lambda = \langle\xi_\lambda|\xi_\kappa\rangle$ describing a spin of the system. A convolution of the vector s^λ with matrixes σ_λ is situated on the right side in Eq. (64). It follows from here, that in s-representation $P = P(s)$ is determined by the expression

$$P(s) = \frac{1}{2}(1 + \gamma_5 s^\mu\gamma_\mu) \qquad s^\mu = \{0, s^\lambda\}. \tag{65}$$

The formula (65) is the known expression for the projection operator selecting spinors with a given orientation in rest. After left multiplication on $\langle x_\lambda^R|$ the requirement of orthonormal basis $\langle x_{\lambda_i}^R|x_{\lambda_j}^R\rangle = \delta_{ij}$ allows us to rewrite the expression (63) in rest in the form

$$\langle x_\lambda^R|x_\lambda^L\rangle\gamma_0\langle x_\lambda^L|\psi\rangle = \langle x_\lambda^R|\xi_\kappa\rangle(\gamma_5\sigma_\lambda)_{\kappa\nu}\langle\,\xi_\nu|x_\lambda^L\rangle\gamma_0\langle x_\lambda^L|\psi\rangle \tag{66}$$

Let us introduce the designations

$$x_\lambda = \langle x_\lambda^R | x_\lambda^L \rangle; \qquad \xi_\kappa = \langle x_\lambda^R | \xi_\kappa \rangle. \tag{67}$$

Taking into account the expression (67), after simple transformations the expression (66) is written as

$$x_\lambda = \xi_\kappa^\dagger \left(\sigma_\lambda \gamma_5\right)_{\kappa\nu} \xi_\nu \tag{68}$$

Now we can calculate the square $|x|^2 = x^\lambda x_\lambda$ of module of the expression (68). Since

$$x^\lambda = \xi_\kappa^\dagger \left(\gamma_5 (\sigma_\lambda^T)^\dagger\right)_{\kappa\nu} \xi_\nu. \tag{69}$$

we have:

$$|x|^2 = \begin{pmatrix} \xi_s \\ \xi_{\dot s} \end{pmatrix}^\dagger \sum_{i=1}^{3} \gamma_5 (\gamma_i^T)^\dagger \gamma_i \left[\xi^\kappa (\xi^\dagger \gamma_5)_\kappa\right] \begin{pmatrix} \xi_s \\ \xi_{\dot s} \end{pmatrix} = |\xi|^2 \begin{pmatrix} \xi_s \\ \xi_{\dot s} \end{pmatrix}^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_s \\ \xi_{\dot s} \end{pmatrix} = |\xi|^4. \tag{70}$$

Here $|\xi|^2 = \sum_{\nu=1}^4 \xi_\nu^2$, Dirac representation of matrices γ^μ, $\mu = 0, 1, 2, 3$ is chosen. It follows from here that the bispinors representation is two-valued one.

5 Conclusion

Relativistic invariant projectors of states in a complex bispinor space on a complex spinor space are constructed. The last allows to find an expression for sections of bundle with connection on group $\mathsf{SU}(4)$ in an explicit form. Within the framework of the proposed geometrical approach the rule of summation over polarizations of states in a complex bispinor space has been derived. It has been shown that states in a complex bispinor space always describe a pair of Dirac's particles.

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